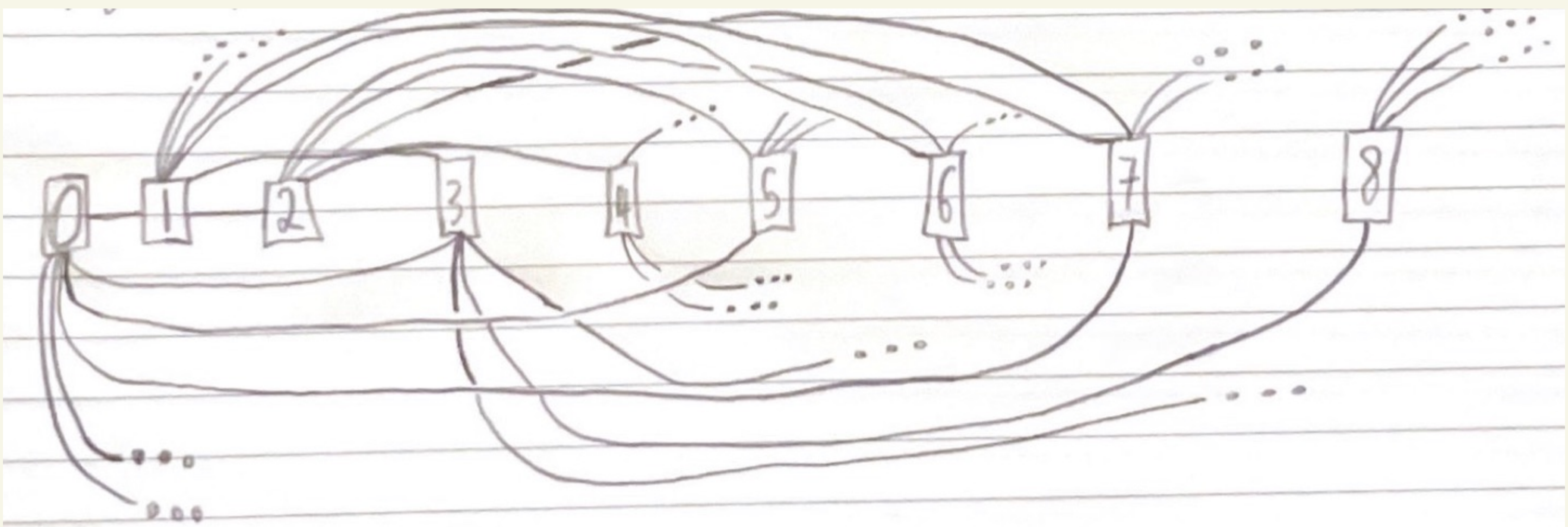
Rado (Erdös-Rényi-Ackerman) graph

• **Def 1:** (Erdös-Rényi, 1963) $G(\infty, \frac{1}{2})$, the* graph produced as a result of connecting pairs of vertices with an edge with probability $\frac{1}{2}$.
↳ $G(\infty, p)$ is an infinite random graph on countably many vertices, with pairs of vertices being connected by an edge with probability $p \in [0, 1]$.
↳ $\forall p$, this seemingly random process yields the Rado graph, R.

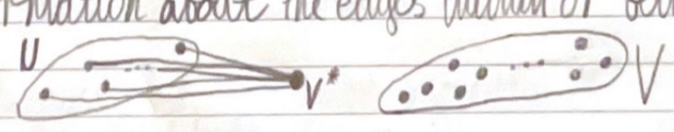
• **Def 2:** (Ackerman-Rado) $V(G) = \mathbb{Z}_0^+$, $E(G)$ = pairs of vertices joined if they satisfy BIT.
↳ "BIT predicate" (Ackerman, 1937) Given non-negative integers a & b, $a < b$, $BIT(b, a) = True$ iff a^{th} bit (binary) of $b \neq 0$.
↳ A vertex n has an edge to any $v^* \equiv 2^{n-1}, \dots, 2^{n-1}-1$ (modulo 2^n)
↳ eg1) take $a = 3$ & $b = 12$, $BIT(12, 3) = T$ since 12 is 1100 in binary.
eg2) take $a = 0$ & $b = 2k$, $BIT(b, 0) = F$ since $\forall 2k$, the least significant digit in binary $= 0$.
eg3) $a = 0$ & $b = 2k+1$, $BIT(b, 0) = T$ since $\forall 2k+1$, the least significant digit in binary $= 1$.



• **Def 3):** Take the set of primes $\equiv 1 \pmod 4$, P, as the set of vertex elements. Join $x, y \in P$ by an edge if $\left(\frac{x}{y}\right) = 1$ [x is a quadratic residue modulo y].
↳ By quadratic reciprocity this happens when $\left(\frac{y}{x}\right) = 1$.
↳ This def. was given by Cameron, 1997.

• **Thm 1)** Let G_1 & G_2 be countably infinite graphs with the "extension property". Then G_1 and G_2 are isomorphic.
↳ This will allow us to prove that the above 3 def.s are equivalent ($\cong$).

• **Def:** Given two finite disjoint collections of vertices $U, V \subset R$, $\exists$ a vertex $v^* \in R - (U \cup V)$ which is connected by an edge to every vertex in U and does not share an edge with any vertex in V. [v^* can be called a 'witness to extension' for (U, V).]
↳ Note: this does not give any information about the edges within or between U and V, we only care about them not sharing any vertices.



• Verifying the Extension property for Def.s 1, 2, and 3 [finding V^*, $\exists$]

↳ For def 1; Given U & V disjoint and finite, each vertex outside of $(U \cup V)$ has an independent $1/2^{|U|+|V|}$ chance of witnessing extension for (U, V). We have a choice of infinitely many vertices $\Rightarrow \exists$ such a vertex.

↳ For def 2; Given $U = \{m_1, m_2, \dots, m_k\}$ and $V = \{n_1, n_2, \dots, n_\ell\}$, take $V^* = \sum_{i=1}^{k} 2^{m_i}$. This V^* has 1's in every spot for U and none for V.

↳ eg) $U = \{0,1,3\}$ and $V = \{2,4,5\}$, then $V^* = 2^0 + 2^1 + 2^3 = 11 \Rightarrow 1011 \rightarrow \underline{0}\underline{0}1\underline{0}11$

↳ For def 3; Let $A, B \subset P = \{\text{primes} \equiv 1 \pmod 4\}$ $A \cap B = \emptyset$ and both finite.
$A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$, let $d = 4a_1a_2\dots a_m b_1 b_2 \dots b_n$. (CRT) & (Dirichlet's Theorem) $\Rightarrow \exists V^*$.

• Proof of Theorem 1:) "Every countable graph w. the extension property is isomorphic."
↳ strategy = try to build an isomorphism inductively, using a technique called "back and forth" to get $f: G_1 \leftrightarrow G_2$.

1) Base case $\Rightarrow$ let $f_0 = \emptyset$ (empty map).
2) Assume we have an isomorphism between some induced subgraphs of G_1 & G_2 and call it f_n.
3) Split the induction into case 1 (n even) and case 2 (n odd) and alternate between them to extend f_n to an isomorphism $f: G_1 \leftrightarrow G_2$ by artificially adding cases to the range and domain (respectively) of f_n.

↳ case 1) n even, let m be the smallest index s.t. x_{m+1} is not in domain(f_n) but x_m is. Define $A, B \subset$ domain$(f_n) \subset G_1$ by $A = \{\text{neighbours of } x_{m+1}\}$ and $B = \{\text{non-neighbours}\}$. G_2 satisfies extension property $\Rightarrow \exists y \in G_2$ which is adjacent to every vertex in $f_n(A) \subset G_2$ and none in $f_n(B)$.
Take such a y to be $f_{n+1}(x_{m+1})$.

↳ case 2) n is odd, let m be the smallest index s.t. y_{m+1} is not in range(f_n) but y_m is. Define $A^*, B^* \subset$ range(f_n) by $A^* = \{\text{neighbours of } y_{m+1}\}$ and $B^* = \{\text{non-neighbours}\}$. G_1 satisfies extension property $\Rightarrow \exists x \in G_1$ adjacent to every vertex of $f_n^{-1}(A^*)$ and isn't joined by an edge to any vertex of $f_n^{-1}(B^*)$.
→ Take such an x to be $f_{n+1}(x) = y_{m+1}$.

4) Take $f = \bigcup_{n \geq 0} [f_n]$ with f being an isomorphism since all of the f_n are.
$f: G_1 \leftrightarrow G_2$, 1-to-1 and onto. domain$(f) = V(G_1)$, range$(f) = V(G_2)$.

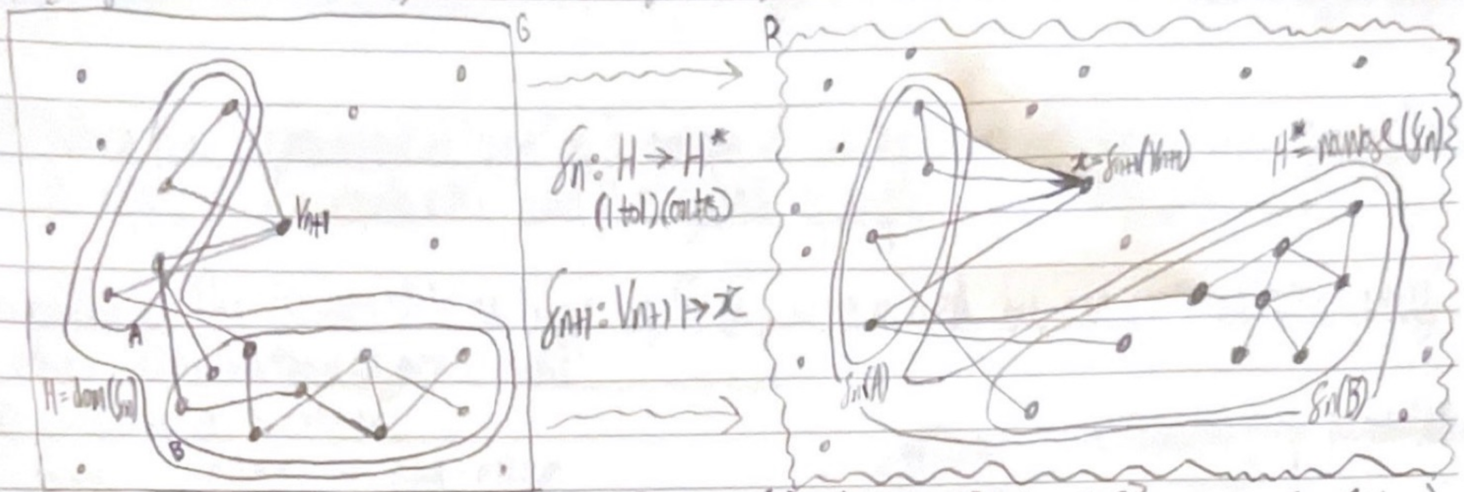
Note about e.p. $\Rightarrow$ not finite

• **NB:)** R is self-complimentary because the extension property is.
↳ This is most easily seen in the probabilistic def. of R.

• **Thm 2)** Every finite or countable graph arises as an induced subgraph of R. "Universality"
{↳ The strategy for this proof is the same as for Thm 1...}

Proof of Thm 2: 1) Base case $f_0 = \emptyset$. Take any finite or countable graph G, with $V(G) = \{v_1, v_2, \ldots\}$. $H = A \cup B \subset G$, $A \cap B = \emptyset$, $H^* = f_n(H) = f_n(A) \cup f_n(B) \subset R$. $f_n : H \to H^*$ (isomorphism)

2) $A = \{$neighbours of v_{n+1} in $H\}$ $B = \{$non-neighbours$\}$



3) Pick $x \in R$ to have $f_n(A) = \{$neighbours$\}$ & $f_n(B) = \{$non-neighbours$\}$; $x := f_{n+1}(v_{n+1})$.

4) Take $f = \bigcup_n f_n \rightsquigarrow$ isomorphism. Above, $f = f_n \cup f_{n+1}$ ∴ domain $f = H \cup \{v_{n+1}\}$ & range$(f) = H^* \cup \{x\}$
range$(f) = f_n(A) \cup f_n(B) \cup f_{n+1}(v_{n+1})$.

• **A Consequence of Thm 2:)**
↳ R contains infinite cliques or cocliques.

• **Def:)** A clique is a complete subgraph.
↳ A coclique is the complement of a clique ∴ a subset of vertices no pair of which are adjacent.

• **Def:)**
A "maximal" clique = a clique that cannot be extended by adding an adjacent vertex.
↳ "maximal" = not contained in any other clique.
↳ no finite clique can be maximal in R (same for cocliques)

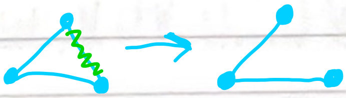
↳ R has infinite maximal cliques & cocliques.
↳ **Eg:)** Enumerate elements of $V(R)$ as $\{v_1, v_2, \ldots\}$ and build a set S by $S_0 = \emptyset$, $S_{n+1} = S_n \cup \{v_m\}$ where m is the least index of a vertex joined to every vertex in S_n, and then let $S = \bigcup_{n \geq 0} S_n$.
This yields an S which is an infinite maximal clique in R.
↳ The complement of S is a maximal coclique in R.

• Partition Regularity of R:

• Theorem 3:) For any finite partition of the vertices of R, the induced subgraph of one cell of the partition is isomorphic to R.

• Theorem 4:) The only countable "partition-regular" graphs are the complete graph, R, and the null graph.
→ Only these 3 graphs have the property that any finite vertex coloring will produce a color whose induced subgraph is isomorphic to the original graph(s).

• Def:) "Switching," with respect to a set of vertices, X, means flipping all edges and non-edges, going between X and its complement.

• Def:) "flipping" an edge ⇒ edge ↔ non-edge. 

• Proposition 1:) Let $A, B \subset R$ be finite disjoint subsets of vertices. Let E be the set of vertices in R which witness the extension property for (A, B).
→ E induces a subgraph $\cong R$!
→ proof: $A, B \subset R$; $A \cap B = \emptyset$
1) $E \subset R$, $E = \{v \in R \mid v$ witnesses extension property for $(A,B)\}$, let the subgraph induced by E be $G \subset R$.
WTS G has the extension property $\therefore \cong R$. (and)

2) Let $A', B' \subset E$ be finite and disjoint.
3) Pick $x \in E$ that is adjacent to all of $A \cup A'$ and none of $B \cup B'$.
4) Then G has the extension property since $x \in G$ witnesses extension for (A', B').

Note about 'strength' of e.p.

• NB:) When the "isomorphism type" of a graph is unchanged as some transformation is applied, the original and the final graphs are isomorphic.

• Proposition 2:) The isomorphism type of R is unchanged by an application of any of the following transformations:
→ a) Deleting a vertex.
b) Flipping an edge or non-edge.
c) Switching w. respect to a finite set of vertices.

} WTS that the extension prop. remains.

Note about adding vertices...

"Robustness"

↳ proof:)

1) Prop. 1 $\Rightarrow \exists$ infinitely many witness vertices in R.

2) $\boxed{a}$ When deleting a vertex we only need to worry about deleting the witness for extension. But, if this happens, by 1) we can always pick an alternative witness.

3) $\boxed{b}$ When flipping an edge or non-edge, the only concern is that we have tampered w. an edge for the witness for extension. But, 1) $\Rightarrow$ we can always find an alternative witness for extension (not affected by edge flip).

4) $\boxed{c}$ Let X be the finite set of vertices with respect to which we switch. $A, B \subset R$ with $A \cap B = \emptyset$ and both finite.
After the switch, picking V^* to be a witness for extension to $((A-X) \cup (B \cap X), (B-X) \cup (A \cap X))$ corrects for switching.

$\therefore$ The extension property for R is unaffected by any of the above transformations.

• Theorem 3 Proof:) "Any finite partition of the vertices of R will have one cell whose induced subgraph has the extension property."

↳ Suppose you have a finite partition of the vertex set, $V_1 \cup V_2 \cup \ldots \cup V_K$, with no cell, V_i, which has the extension property.

$\Rightarrow \forall i, \exists$ finite disjoint subsets $A_i, B_i \subset V_i$, s.t. $\nexists$ vertex $\in V_i$ that is "correctly joined" to all vertices of A_i and not joined to any of B_i.

$\Rightarrow$ Taking $A = A_1 \cup A_2 \cup \ldots \cup A_K$ and $B = B_1 \cup B_2 \cup \ldots \cup B_K$ we have $A, B \subset R$ finite disjoint subsets for which $\nexists$ vertex that is a witness for extension ⨳

⨳ This is a contradiction since R must have the extension property.

• NB:) (Bonato-Sauer, 1996) Given a finite edge-colouring of R, there is a subgraph of R using only 2^* colors of edges that is isomorphic to R. [*1 may not be enough with edges].

• Homogeneity of R:)

↳ Any isomorphism between finite induced subgraphs of R can be extended to the whole of R.

↳ By above, we can get automorphism for R from isomorphism of finite induced subgraphs, this property characterises R up to isomorphism (like the extension prop.).

• Fun fact:) Aut (R) is a subgroup of the homeomorphism group of $\mathbb{R}$.

• The Rado Simplicial Complex:

↳ Def: A simplicial complex X is a set of vertices $V(X)$ and a set of non-empty finite subsets of $V(X)$, called simplices, s.t. any vertex $v \in V(X)$ is a simplex $\{v\}$, and any subset of a simplex is a simplex.

↳ X is said to be countable / finite if its vertex set $V(X)$ is countable / finite.

↳ The high-dimensional generalisation of the Rado graph since the 1-skeleton of the Rado S.C. is R.

↳ The Rado S.C. has countably many vertices.

↳ Analogue of Thm 2 = 'Any countable simplicial complex is an induced subcomplex of the Rado S.C.'

↳ Also, homogeneous = 'Any 2 isomorphic finite induced subcomplexes are related by an automorphism of the whole S.C.'

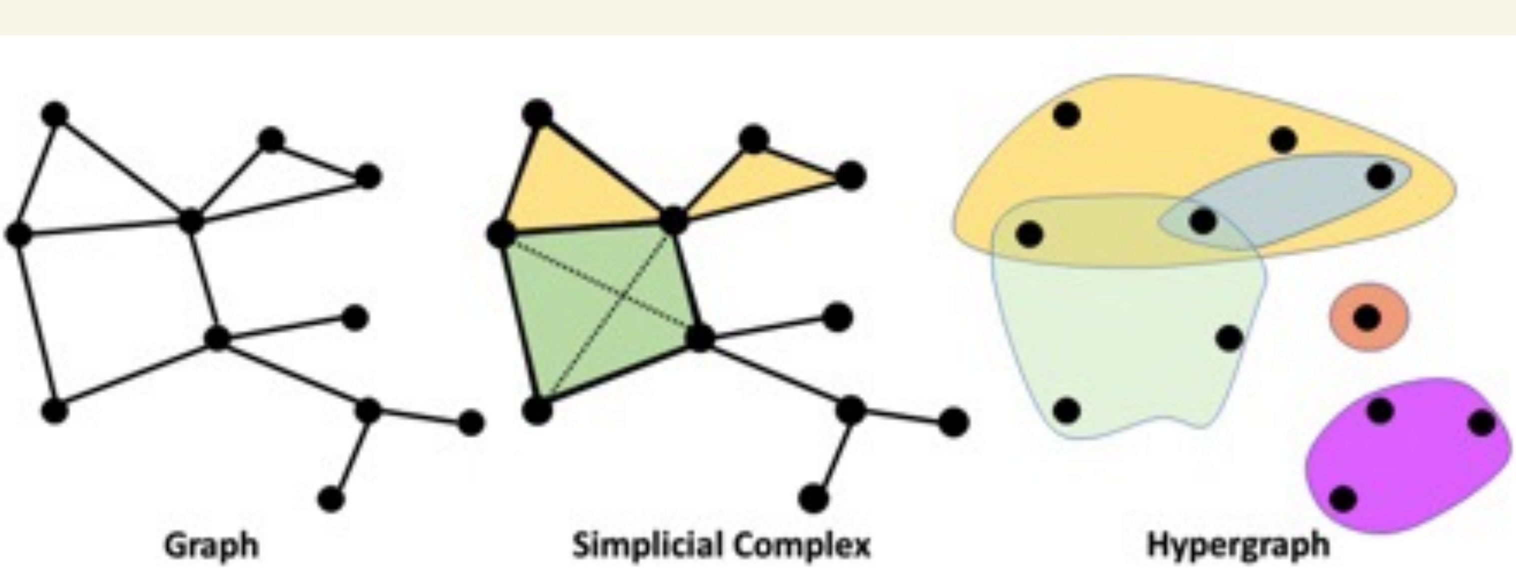
↳ Homogeneity and the adapted version of Thm 2 characterise the R.S.C up to isomorphism. (There is also an analogue of the extension prop. = 'ampleness')

↳ Theorem 3 also has an analogue for the R.S.C. = 'any finite partition of $V(X)$ of the R.S.C. will have at least one cell whose induced S.C. is isomorphic to the R.S.C.'

↳ Property 2 also has an analogue = 'robustness' = removing any finite set of simplexes leaves a S.C. isomorphic to the R.S.C.

↳ High-dim. approximations of the Rado S.C. may be stable* / partition nicely due to the properties above (robustness & ampleness).

• Every simplicial complex is a hypergraph when every simplex (except the empty simplex) is treated as a hyperedge.



• Applications of the Rado S.C.:

↳ Used for modelling complex networks of many objects, whose interactions can occur in groups of 2 or more objects.

↳ pairwise interactions can be recorded by representing the system as a graph but higher order interactions require including simplicial complexes of $d \geq 2$

↳ such networks appear in neuroscience, ecology, biochemistry, and in the study of social systems.

• A Fun Fact: The Rado S.C. is isomorphic to a triangulation of the simplex $\Delta_{\mathbb{N}}$.

↳ The 'Geometric realisation' of the Rado complex is homeomorphic to the geometric realisation of the infinite dimensional simplex $|\Delta_{\mathbb{N}}|$.